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Note

The g-theorem matrices are totally nonnegative

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ABSTRACT

The g-theorem proved by Billera, Lee, and Stanley states that a sequence is the g-vector of a simplicial polytope if and only if it is an M-sequence. For any d-dimensional simplicial polytope the face vector is gM_d where M_d is a certain matrix whose entries are sums of binomial coefficients. Björner found refined lower and upper bound theorems by showing that the (2×2)-minors of M_d are nonnegative. He conjectured that all minors of M_d are nonnegative and that is the result of this note.

1. Introduction

Of all kinds of polytopes those with only simplicial faces, the simplicial polytopes, are the most well-understood ones. The possible number of faces in different dimensions is completely determined by the g-theorem [3,4,11]. It gives a linear relation between f-vectors (which contains the number of faces by dimension) and sequences characterized by Macaulay [9] algebraically. In this note we prove a conjecture by Björner on the matrices M_d that define these linear relations. Namely, we prove that all their quadratic submatrices have nonnegative determinants, that is, the matrices M_d are totally nonnegative.

For a d-dimensional polytope let f_i be the number of its i-dimensional faces and let its f-vector be $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$ with $f_{-1} = 1$. If the polytope is simplicial we will also associate the g-vector $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ with it. The main property of the g-vector is that it is an M-sequence. There are many ways to define M-sequences using concepts from different parts of mathematics, and in Chapter 8 of Ziegler's book on polytopes [12] there is an excellent description of them. The only thing close to a definition of them in this note is implicit in the linear relation and the g-theorem.

Let M_d be the matrix defined by

$$(M_d)_{\substack{0 \leqslant i \leqslant \lfloor d/2 \rfloor \\ 0 \leqslant j \leqslant d}} := \binom{d+1-i}{d+1-j} - \binom{i}{d+1-j}.$$

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For example

$$M_3 = \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 3 & 2 \end{pmatrix}.$$

The relation between f and g-vectors is $f = gM_d$. Björner [5] proved that all (2×2)-minors of M_d are nonnegative and used that to prove a comparison theorem for f-vectors with corollaries that refined the lower and upper bound theorems. He conjectured that all minors of M_d are nonnegative, in other words, that the M_d matrices are totally nonnegative. This was confirmed for $d \leqslant 13$ by A. Hultman. Theorem 2.1 states that Björner's conjecture is true.

2. Lattice paths and nonnegative minors

A path from (x_1, y_1) to (x_2, y_2) in $\mathbb{Z}^2$, where $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, is called a lattice path if only the steps $(1,0)$ and $(0,1)$ are allowed. The number of lattice paths from $(0,0)$ to (m,n) which do not touch the line $y = x + t$ is $\binom{m+n}{n} - \binom{m+n}{n-t}$ if $t > 0$ [2,10]. These numbers are sometimes referred to as the ballot numbers. The weight of a path is the product of the weights of its arcs. From any subset A of $\mathbb{Z}^2$ one can construct a planar acyclic directed graph with vertex set A and arcs of types $(x, y) \to (x+1, y)$ and $(x, y) \to (x, y+1)$.

Theorem 2.1. *The matrix M_d is totally nonnegative for all d.*

Proof. For integers $d \geqslant 1$ define the graphs T_d with vertex set

$$\left\{(x, y) \in \mathbb{Z}^2 \;\middle|\; x \leqslant \lfloor d/2 \rfloor \leqslant x + y \text{ and } y - x \leqslant \lceil d/2 \rceil\right\}.$$

The weight of horizontal arcs in T_d is 1, and the weight of any vertical arc $(x, y) \to (x, y+1)$ is w_y. The graphs T_7 and T_8 are drawn in Fig. 1. For all $0 \leqslant i \leqslant \lfloor d/2 \rfloor$ and $0 \leqslant j \leqslant d$, let the sum of the weights of the directed paths from $(\lfloor d/2 \rfloor - i, i)$ to $(\lfloor d/2 \rfloor, j)$ be $W(d, i, j)$. If $i > j$ then $W(d, i, j) = 0$ and otherwise it is

$$w_i w_{i+1} \cdots w_{j-1} \left(\binom{j}{i} - \binom{j}{d+1-i} \right)$$

by the ballot numbers.

Now define the weights of the vertical arcs as $w_i = \binom{d+1}{i+1} / \binom{d+1}{i}$. Note that all arc weights are positive real numbers. For $i \leqslant j$ we get that

$$\begin{aligned} W(d, i, j) &= \frac{\binom{d+1}{i+1}}{\binom{d+1}{i}} \frac{\binom{d+1}{i+2}}{\binom{d+1}{i+1}} \cdots \frac{\binom{d+1}{j}}{\binom{d+1}{j-1}} \left(\binom{j}{i} - \binom{j}{d+1-i} \right) \\ &= \binom{d+1}{j} \binom{d+1}{i}^{-1} \binom{j}{i} - \binom{d+1}{j} \binom{d+1}{i}^{-1} \binom{j}{d+1-i} \\ &= \binom{d+1-i}{d+1-j} - \binom{i}{d+1-j}. \end{aligned}$$

Thus, M_d is just

$$W(d, i, j)_{\substack{0 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor \\ 0 \leqslant j \leqslant d}} = \begin{pmatrix} \binom{d+1}{d+1} - \binom{0}{d+1} & \binom{d+1}{d} - \binom{0}{d} & \binom{d+1}{d-1} - \binom{0}{d-1} & \cdots & \binom{d+1}{1} - \binom{0}{1} \\ 0 & \binom{d}{d} - \binom{1}{d} & \binom{d}{d-1} - \binom{1}{d-1} & \cdots & \binom{d}{1} - \binom{1}{1} \\ 0 & 0 & \binom{d-1}{d-1} - \binom{2}{d-1} & \cdots & \binom{d-1}{1} - \binom{2}{1} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & & \binom{d+1-\lfloor d/2 \rfloor}{1} - \binom{\lfloor d/2 \rfloor}{1} \end{pmatrix}.$$

Fomin and Zelevinsky wrote a nice survey on testing total positivity and related questions [6]. We need a result by Lindström [8], and Gessel and Viennot [7]. There is also a beautiful exposition of it in [1].

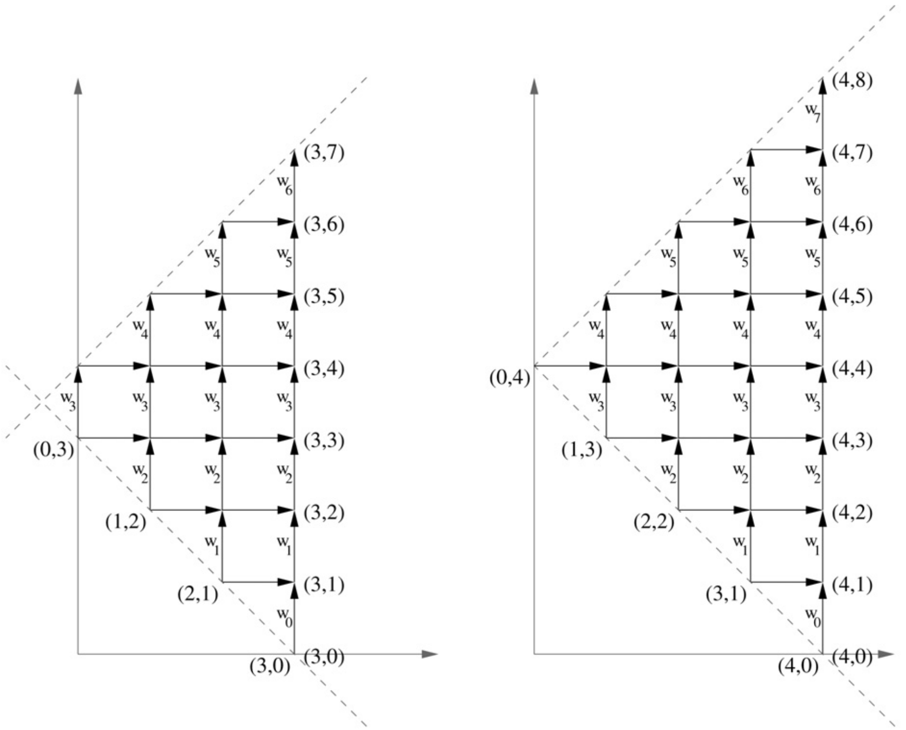

Fig. 1. The weighted planar directed graphs T_7 and T_8. The starting points are on the negative sloped line and the end points are on the vertical boundary line.

Lemma 2.2. *Let T be a nonnegatively weighted planar directed acyclic graph with the vertices $a_k, a_{k-1}, \dots, a_1, b_1, b_2, \dots, b_l$ on the boundary in that order. Then the $(k \times l)$-matrix with the sum of the weights of the paths from a_i to b_j as (i, j)-element is totally nonnegative, since its $(m \times m)$-minors count the weighted number of families of m nonintersecting lattice paths with appropriate starting and end points.*

Using the lemma with T_d as T gives that M_d is totally nonnegative. □

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